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A note on non-smooth programming problems

Mohammad Mehdi Mazarei *, Ali Vahidian Kamyad, Ali Asghar Behroozpoor

Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, International Campus, Mashhad, Iran

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ABSTRACT

In this paper, we introduce a new approach to obtain a novel numerical solution of nonlinear programming problems (NLP) which the objective function (functions) or constraint function (functions) are non-smooth ones. This technique is based on a new piecewise linearization approach. In fact, we transfer the nonlinear programming problem (NLP) to a variational problem that would reduce the new approximated problem to a linear programming problem (LP). Then, the approximated solution of the original problem would be obtained by the LP problem. Finally, numerical examples are given to show the efficiency of the proposed approach.

1. Introduction

A wide range of problems arising in practical applications can be formulated as nonlinear programs (NLPs). This includes chemical engineering (Grossmann and Sargent, 1979; Corsano et al., 2011), water network problems (Misener and Floudas, 2012), gas (Bragalli et al., 2006), energy (Murray and Shanbhag, 2006), time-loss in the paper industry (Harjunkoski et al., 1999), concrete structure design (Guerra et al., 2011), load-bearing thermal insulation systems (Abhishek et al., 2010), medical applications (Pardalos et al., 2004) and so on. There are some methods and algorithms available in the literature for the case in which the objective and constraint functions are convex and differentiable, e.g. the optimal point can be obtained using Karush-Kuhn-Tucker conditions. Furthermore, the penalty and barrier methods can be used for constraint optimization problems. Although, some other methods based on linear programming exist, such as the method of approximate programming (Griffith and Stewart, 1961; Kamyad et al., 2005). After Han proved local and global convergence of SQP methods in (Han, 1976; Han, 1977), a large amount of research papers have been produced on SQP-based techniques. However, all of the aforementioned techniques are for differentiable (smooth) or convex problems and some optimality conditions are necessary such as continuity, convexity and differentiability. But, many important practical problems are naturally modeled as non-smooth NLPs and these methods are inadequate to solve them.

The non-smooth NLPs would be useful in much application of sciences. Some investigations have been done in non-smooth programming problems (Rockafellar, 1994; Nesterov, 2005). However, these techniques are not efficient for non-convex non-smooth optimization problems. It is worthwhile to mention that many well-studied optimization problems can also be naturally viewed as non-convex and non-smooth NLPs. In this study, we have proposed a new technique to solve nonlinear non-smooth programming problems.

The rest of this paper is organized as follows. In next section, we introduce a technique to find the best piecewise linearization of nonlinear functions. In section 3, we explain the equivalency of non-linear and linear programming problem. In Section 4, we illustrate some numerical examples to demonstrate the efficiency and accuracy of the proposed approach.

2. A new piecewise linearization of nonlinear function

The linearization of nonlinear systems is an efficient tool for finding approximate solutions and treatment analysis of these systems.

Remark 2.1: Since, every non-smooth function is a nonlinear function, we consider nonlinear functions.

Let $F: A \subseteq R^n \to R$ be a nonlinear function. We suppose that $x \in A \subseteq R^n$ and the subset A is

* Corresponding Author.
Email Address: mazarei_mehdi@yahoo.com (M. Mazarei)
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compact. Our aim is to approximate the nonlinear function F by a piecewise linear function as follows (Eq. 1):

$$F_N(x) = \sum_{i=1}^{N}(a_{i0} + a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)X_{A_i}(x) \quad (1)$$
$$a_{ij} \in R; i = 1, 2, \dots, N$$

where, A_i is ith subset in partitioning of A as $P_N = \{A_1, A_2, \dots, A_N\}$. As we know, this partitioning has the bellow properties:

1) $\forall i, j = 1,2, \dots, N;\ A_i \cap A_i = \emptyset; A_i \in R^n, A_i \neq \emptyset$
2) $A = \cup_{i=1}^{N} A_i$

also, A_i is Lebesgue measurable set and $X_{A_i}(x)$ be the characteristic function on A_i defined as follows (Eq. 2):

$$X_{A_i}(x) = \begin{cases} 1, & x \in A_i \\ 0, & x \notin A_i \end{cases} \quad (2)$$
$$L_1(A) = \left\{f: A \to R \mid \int_A |f|dx < \infty\right\}$$
$$L_2(A) = \left\{f: A \to R \mid \int_A |f|^2dx < \infty\right\}.$$

Now, let and them. As we know $L_2(A)$ is a Hilbert space of A with the following inner product (Eqs. 3 and 4):

$$\langle f, g\rangle = \left(\int_A f(x)g(x)dx\right)^{1/2}; f, g \in L_2(A) \quad (3)$$
and
$$\|f\|_2 = \left(\int_A |f|^2dx\right)^{1/2} \quad (4)$$

Definition 2.1: We define $S_N(A)$; $(N \in N)$ be the set of all $F_N \in L_1(A)$ of the form (1).

Lemma 2.1: For 0<P<q we have $L_P \subset L_q$.

So, we have $L_1 \subset L_2$. We use the norm-1 in the rest of paper.

Definition 2.2: If $F: R^n \to R$ is a nonlinear function and $F_N \in S_N(A)$, we define $\|F - F_N\|_1$ as follows (Eq. 5):

$$\|F - F_N\|_1 = \int_A |F - F_N|dx \quad (5)$$

Lemma 2.2: The subset $S_N(A)$ is dens on $L_1(A)$.

Proof: Suppose that F is a nonlinear function that,
$F: A \subseteq R^n \to R$
$\forall \varepsilon > 0 \exists N(\varepsilon) \in N, \exists F_N(.) \in S_N(A)$
$\|F - F_N\|_1 < \varepsilon$.

Definition 2.3: We call $F^* \in S_N(A)$ the best piecewise linear approximation of F if for any $F_N \in S_N(A)$ we have,

$$\|F - F^*\|_1 \leq \|F - F_N\|_1$$

therefore, F^* is the optimal solution of the following optimization problem (Eq. 6).

$$Min\|F - F_N\|_1$$
$$F \in S_N(A) \quad (6)$$

Obviously, because $0 \leq \|F - F_N\|_1$, the optimization problem (6) has optimal solution.

To clarify our approach, first we consider a nonlinear function $F: R \to R$. Second, we explain this approach for a nonlinear function $F: R^n \to R$.

(i) Consider the optimization problem

$$Min\|F - F_N\|_1$$
$$f \in S_N(A)$$

where, $F: A \subseteq R \to R$ is a nonlinear function and $A = [a, b]$. As we know $[a, b]$ can be replaced by$[0, 1]$.

Now, we decompose interval $[0, 1]$ to N subintervals $\left[\frac{i-1}{N}, \frac{i}{N}\right]$; $1, 2, \dots, N$ (Fig. 1). Since, $F_N \in S_N(A)$, we have (Eq. 7)

$$Min \int_0^1 \left|F(x) - \sum_{i=1}^{N}(a_i + b_ix)x_{\left[\frac{i-1}{N}, \frac{i}{N}\right]}(x)\right| dx \quad (7)$$

Our objective function is a functional. Now, we reduce this functional to a summation as follows (Eq. 8):

$$\int_0^1 \left|F(x) - \sum_{i=1}^{N}(a_i + b_ix)x_{\left[\frac{i-1}{N}, \frac{i}{N}\right]}(x)\right| dx \cong \frac{1}{N}\sum_{i=1}^{N}|F(x_i) - (a_i + b_ix)| \quad (8)$$

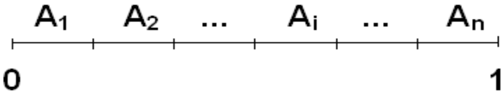

Fig. 1: Partitioning A to subintervals A_i.

so, the optimization problem (8) is as follows (Eq. 9):

$$Min \sum_{i=1}^{N}|r_i + s_i|$$
$$r_i - s_i = F(x_i) - (a_i + b_ix_i)$$
$$0 \leq r_i, s_i; i = 1, 2, \dots, N \quad (9)$$

But, the optimization problem (9) is a nonlinear programming problem. We reduce this problem to a linear programming problem by relation $|r_i - s_i| = r_i + s_i$. So, our optimization problem will be as the following linear programming problem (Eq. 10):

$$Min \frac{1}{N}\sum_{i=1}^{N} r_i + s_i$$
s.t
$$r_i - s_i = F(x_i) - (a_i + b_ix_i)$$
$$0 \leq r_i, s_i; i = 1, 2, \dots, N \quad (10)$$

(ii) Second, we consider a nonlinear function $F: A \subseteq R^n \to R$ then the optimization problem would be as follows:

$$Min \int_A \left|F(x_1, x_2, \dots, x_n) - \sum_{i=1}^{N}(a_{i0} + a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)X_{A_i}(x)\right|dx$$

As, we explained in section (i) this optimization problem will be reduced to a linear programming problem as follow (Eq. 11):

$Min \sum_{i=1}^{N} r_i + s_i$
s.t
$r_i - s_i = F(x_1, x_2, \dots, x_n) - (a_{i0} + a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n)$
$0 \le r_i, s_i; i = 1, 2, \dots, N$
$N = m_1, m_2, \dots, m_n$ (11)

Where, $m_1, m_2, \dots, m_n$ are the numbers of subintervals on axises $x_1, x_2, \dots, x_n$, respectively.

3. Nonlinear programming problem

Now, we consider the nonlinear programming problem as follows:

$Min\ F(x_1, x_2, \dots, x_n)$
s.t
$G_i(x_1, x_2, \dots, x_n) \ge 0; i = 1,2, \dots, m$
$G_k(x_1, x_2, \dots, x_n) = 0; k = m+1, m+2, \dots, l$ (12)

where, the objective function F or constraint functionsG_i, G_k are nonlinear functions. According to the previous section, we replace objective and constraint functions by the best piecewise linear functions of form (1). Therefore, the nonlinear programming problem (12) is approximately equal to a linear programming problem as follows:

$Min\ \sum_{i=1}^{N} \{a_{i0} + a_{i1}x + a_{i2}y\} X_{A_i}(x, y)$
s.t
$$\sum_{i=1}^{N} \{b_{i0} + b_{i1}x + b_{i2}y\} X_{A_i}(x, y) \ge 0; i = 1, 2, \dots, n$$
$\sum_{i=1}^{N} \{c_{i0} + c_{i1}x + c_{i2}y\} X_{A_i}(x, y) = 0; k = m + 1,\ m + 2, \dots, l$ (13)

Now, according to lemma1, we can approximate objective function F and constraint functions G_i and G_k arbitrarily. Then, the objective function and feasible region in nonlinear programming problem (12) is approximated to the objective function and feasible region in linear programming problem (13). This lead to the optimal solution of linear programming problem (12) is the approximated optimal solution of problem (13). This is clear we can improve the accuracy of approximated optimal solution by increasing N.

4. Numerical examples

In this section, we show the efficiency of new technique by some examples.

Example 1: Consider following nonlinear smooth programming problem:
$Min\ e^x$
s.t
$\sin(x) - x \le 1$
$1 \le x \le 3$ (14)

We may convert interval [1, 3] to [0, 1]. For this purpose, we may define bijective function *H(x)* as follows:

$H: [1, 3] \to [0, 1]$
$x \to \frac{x-1}{2}$
Now, we have:
$Min\ e^{(2x+1)}$
s.t
$\sin(2x + 1) - 2x \le 2$
$0 \le x \le 1$ (15)

The optimal solution is $x^* = 3$. We have used the piecewise linearization of objective function and constraint functions to approximate non-smooth programming problem (15) to a linear programming problem. The optimal solution of approximated linear programming problem for n=10, 20, 50, 100 has been showed in Table 1.

Table 1: The approximate optimal point of example 1

n	Approximate optimal solution
10	2.9020
20	2.9813
50	2.9900
100	2.9950

Example 2: Consider following nonlinear non-smooth programming problem:

$Min\ e^{|2x-1|}$
s.t
$\sin(x) \le 1$
$0 \le x \le 1$ (16)

In this example, the objective function is non-smooth function. The exact optimal solution of nonlinear non-smooth programming problem (16) is $x^* = 1$.

We have used the piecewise linearization of objective function and constraint function to approximate non-smooth programming problem (16) to a linear programming problem (Figs. 2 and 3). The optimal solution of approximated linear programming problem for n=10, 20, 50, 100 has been showed in Table 2.

Table 2: The approximate optimal point of example 2

n	Approximate optimal solution
10	0.9471
20	0.9631
50	0.9802
100	0.9943

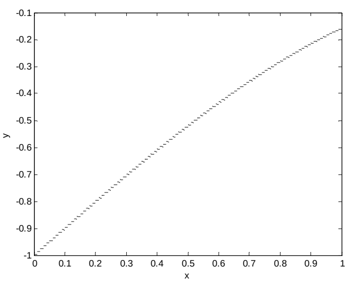

Fig. 2: The piecewise linearization of constraint function of example 2 (n=100).

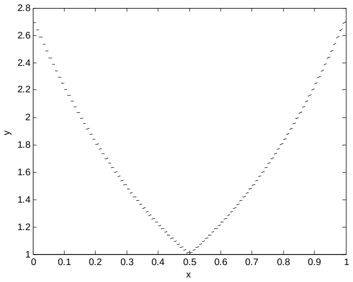

Fig. 3: The piecewise linearization of objective function of example 2 (n=100)

Example 3: Consider following nonlinear non-smooth programming problem:

$$\begin{aligned} &Min\ \ |x - y^3| \\ &\text{s.t} \\ &y - x^2 \geq 0 \\ &y + x - 2 \geq 0 \\ &0 \leq x, y \end{aligned} \qquad (17)$$

In this example, the objective function is nonlinear non-smooth function. The exact optimal solution of nonlinear programming problem (17) is $(x, y)^* = (1, 1)$.

We have used the piecewise linearization of objective function and constraint functions to approximate non-smooth programming problem (17) to a linear programming problem. The optimal solution of approximated linear programming problem for n=10, 20, 50, 100 has been showed in Table 3.

Example 4: Consider following nonlinear non-smooth programming problem:

$$\begin{aligned} &Min\ \ x^2 + y^2 \\ &\text{s.t} \\ &y + x^2 - 1 \leq 0 \\ &y - |x - 1|^3 \geq 0 \\ &0 \leq x, y \end{aligned} \qquad (18)$$

Table 3: The approximate optimal point of example 3

n	Approximate optimal solution
10	(0.8103, 1.172)
20	(0.9326, 1.0740)
50	(0.9720, 1.0337)
100	(0.9874, 1.0025)

In this example, one of constraint functions is non-smooth function. The exact optimal solution of nonlinear programming problem (18) is $(x, y)^* = (0.35, 0.27)$.

We have used the piecewise linearization of objective function and constraint functions to approximate non-smooth programming problem (18) to a linear programming (Figs. 4 and 5). The optimal solution of approximated linear programming problem for n=10, 20, 50, 100 has been showed in Table 4.

5. Conclusion

We have solved the nonlinear smooth and non-smooth programming problems by introducing a novel technique. Since, the competing methods in the literature need differentiability of objective and constraint functions, none of them is appropriate to solve the non-smooth problems. In this approach, we transfer the nonlinear programming problem to a variational problem. Then, we reduce it to a linear programming problem (LP). By solving the LP programming problem, we obtain the approximated solution of the original problem.

Table 4: The approximate optimal point of example 4

n	Approximate optimal solution
10	(0.3188, 0.2290)
20	(0.3385, 0.2571)
50	(0.3413, 0.2643)
100	(0.3482, 0.2676)

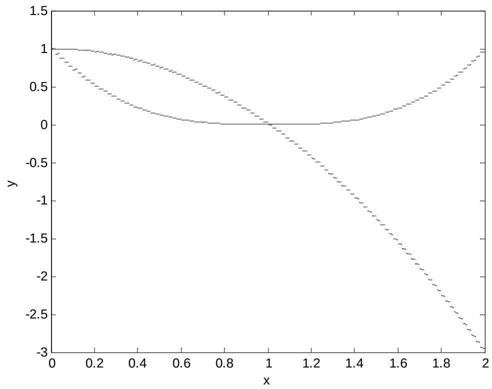

Fig. 4: The piecewise linearization of constraint functions of example 4 (n=100)

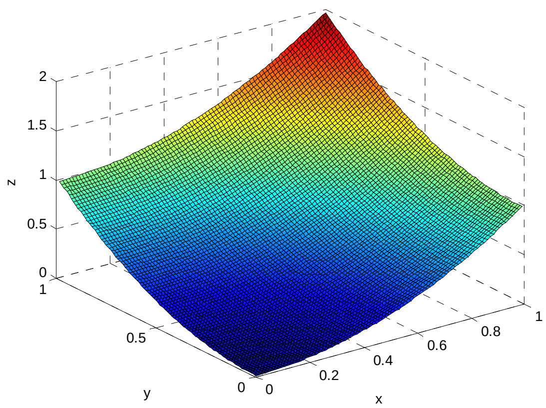

Fig. 5: The piecewise linearization of objective function of example 4 (n=100).

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